$$\frac{.57735^4\left(\frac{.44654^2}{.86603}-.28940^2\right)}{54\times .0000339822}=-26.6 \text{ feet.}$$

Sum of the first and second terms of (3) gives, as the value of x. 6026.5 feet. This result is less than 1 foot in error.

For still greater accuracy either larger values of m may be taken or more than two terms of (A) may be employed.

It is perhaps needless to point out that the same method of evaluating the integrals for y and t may be quite conveniently employed.

OUTLINE OF A COHERENT COURSE IN COLLEGE ALGEBRA.

By DR. A. C. LUNN.

In teaching what may be considered the standard topics in a course on College Algebra it is often difficult to avoid leaving with the students the impression that they have labored over a set of isolated subjects, hard to master because of lack of interrelations. The following outline sketches the result of an attempt to organize many of these subjects into a course, to be developed so that a certain natural unity should be more or less obvious.

§1. The Fundamental Problems.

The main subject is the study of rational functions of a single real variable, in three aspects, the Formula, the Graph, and the Table, denoted in the following by their initial letters. The motto is that every notion, operation, and problem connected with the theory is to be interpreted so far as possible in all three aspects.

By "formula" is understood an algebraic statement of the form $y=f(x)$, giving directions for finding the value of the dependent variable y in terms of the argument x by a certain set of operations, $f(x)$ denoting for the present purpose "an expression in terms of x". The functions or expressions considered here are then either polynomials of the form $A_0x^n+A_1x^{n-1}\ldots\ldots A_{n-1}x+A_n$, or quotients of such, although opportunity is presented for easy passage to certain irrational forms if desired, and certain of the theorems deduced apply to wide classes of functions.

By "graph" is understood a plot of the curve whose equation is $y=f(x)$ in Cartesian coördinates, where the two units of measure are chosen independently according to dictates of convenience for the two axes (not necessarily rectangular), since for present purposes lengths are compared only when parallel.

By a "table" is meant an array of numerical values of y corresponding to assigned values of x, and where not otherwise specified it is understood that the

table is what might be called normal, with the argument values equally spaced; in this case the table includes the columns of differences of various orders as far as found to be needed for completeness.

The primary problems of the theory are then, for a function known in any one of these three aspects, to find means of giving it expression in the other two. This gives six fundamental problems, of which the order of presentation is to some extent arbitrary. The following order is perhaps as good as any.

Problem TG: given the table, to make the graph. This is solved simply by plotting the points given in the table, the inevitable incompleteness of the table being manifest. Practice here is secured well from tables of physical or astronomical observations or statistics, for which no formula is known to the students.

Problem GT: this implies the measurement of chosen ordinates of a given curve, and the insertion of their numerical values in a tabular scheme.

These two problems need be insisted upon only if the idea of plotting is not already somewhat familiar.

Problem FT. For a polynomial this problem is solved for example by the familiar method of synthetic substitution with detached coefficients, implying for each value of the argument an operation such as the following:

$$
\begin{array}{lllll}
x) & A_0 & A_1 & A_2 \ldots\ldots\ldots\ldots\ldots\ldots & A_n \\
 & & A_0x & A_0x^2+A_1x & A_0x^n+\ldots+A_{n-1}x \\
\hline
 & A_0 & A_0x+A_1 & A_0x^2+A_1x+A_2 & A_0x^n+\ldots+A_n
\end{array}
$$

Many other modes of computation are suggested by the various shapes into which the formulas are thrown for other purposes.

Problem FG. This problem might of course be solved indirectly by computing a table and plotting from the table, or in short, resolving the step FG into the steps FT and TG. But it is much better to do it by purely geometric processes, for which the operations or transformations given in §2 are useful. The manifold ways of using these will be better illustrated in special cases.

Problem TF. Here again the incompleteness of the table is brought vividly home, since it is obviously possible to make many formulas fit the same table of the ordinary kind. But special kinds of solution will be noticed later; in particular the determination of a polynomial of degree n which takes assigned values for $n+1$ given values of the argument.

Problem GF. An outlook is afforded here on the vast fields of mathematical research. If the graph is known to be that of a polynomial of a certain degree the coefficients can be determined by algebraic procees. But the determination of an analytic formula for a given graph is a problem on which has been expended an important part of the mathematical endeavor of a century.

The six problems named, in direct or modified form, make the primary elements of the course, about which all other material is grouped, either as direct outgrowth, or as auxiliary.

§2. The Geometrical Operations.

In the graphic construction of curves which are the graphs of rational functions it is convenient to make use of the following geometric constructions or operations.

(1) The x-Push or Translation through distance a (positive or negative). The curve is moved bodily parallel to the x-axis without changing form, size, or orientation in the plane. The effect on the formula may be indicated thus:

$$ {}^{x}P_{a}f(x)=f(x-a), $$

since the value of the ordinate after the operation must be read as the value previously corresponding to an x-value less by a than its new value.

(2) The y-Push:

$$ {}^{y}P_{b}f(x)=f(x)+b. $$

(3) The x-Stretch or Elongation, with factor m:

$$ {}^{x}S_{m}f(x)=f\left(\frac{x}{m}\right). $$

(4) The y-Stretch, with factor n:

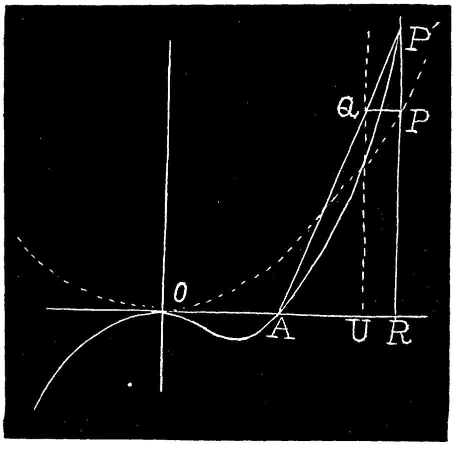
$$ {}^{y}S_{n}f(x)=nf(x). $$

(5) The addition of two curves by algebraic addition of ordinates (operation A).

(6) Multiplication by abscissa difference from assigned point a, where the result is indicated symbolically thus:

$$ M_{a}f(x)=(x-a)f(x). $$

This is accomplished by the following construction,* where the original and the final position of the point are marked P and P'.



A is point of abscissa a; U is point of abscissa $a+1$.

$$ \frac{y'}{y}=\frac{RP'}{RP}=\frac{RP'}{UQ}=\frac{AR}{AU}=x-a, \quad y'=(x-a)y. $$

*Used by Professor Moore in course in Calculus since 1902; see also Peano, *Applicazioni del Calcolo Infinitesimale*, p. 74.

Here P is given and P' required; the converse is also needed, P' being the known, P the sought point. The effect of this transformation on a given curve is illustrated in the figure, where the original curve (dotted), is $y=x^2$, the new curve (full line), $y=(x-2)x^2$, with the value $a=2$.

These six operations, which will be denoted by the respective symbols indicated, are to be employed in the construction of a curve when the formula is assigned. Illustrations may now be given.

§3. Standard Expressions and Their Graphs.

Start with the line $y=1$, parallel to the x-axis; then the operation M_0 gives the "standard diagonal," whose equation is $y=x$; another application of M_0 gives the parabola $y=x^2$, and by repetition are obtained all the curves of the form $y=x^n$, where n is a positive integer.

With a new start at the line $y=1$, the inverse operation of M_0, which may be called M_0^{-1}, gives the hyperbola $y=x^{-1}$, and by repetition all curves of the type $y=x^{-n}$. By the use of any reference point A on the x-axis instead of the origin may be obtained any curve of the forms $(x-a)^n$, $(x-a)^{-n}$. Or these may be obtained by the operation ${}^xP_a$, applied to the curves x^n, x^{-n}.

A modification of this set is important. Start with the graph of any expression $f(x)$, which does not meet the axis or go to infinity at the point a, and obtain by direct and inverse operations the curves $(x-a)^nf(x)$, $(x-a)^{-n}f(x)$. It is obvious from inspection of the graphs that the general character of the curves in the neighborhood of the point a is the same as for the symple types $(x-a)^n$, $(x-a)^{-n}$, of corresponding exponent. This is expressed by saying that the point a is a 'root' of order n of the function $(x-a)^nf(x)$, and a 'pole' of order n of the function $(x-a)^{-n}f(x)$.

This prepares the way for the systematic study of polynomials of various degrees and their quotients.

The straight line $y=A_0x+A_1$ is constructed from the diagonal $y=x$ by the successive operations ${}^yS_{A_0}$, ${}^yP_{A_1}$, and the interpretation of A_0 as slope and A_1 as y-intercept thus made clear, to yield a shorter method of construction with ruler.

Or with the equation written in the form $y=A_0(x-a)$, where $a=-A_1/A_0$, the same line may be gotten from $y=x$ by x-push of value a, and y-stretch of value A_0.

Or finally the line $y=A_0$ may be operated upon by M, with reference-point $a=-A_1/A_0$.

As the degree increases, varieties of construction multiply rapidly; no attempt at completeness need be made here.

In the case of the parabolas,or quadratic functions, for instance, there are among others the four important forms:

$$y=A_0x^2+A_1x+A_2, \qquad y=A_0(x^2+px+q),$$
$$y=A_0[(x-a)^2+b], \qquad y=A_0(x-r_1)(x-r_2),$$

the final form being considered only when the roots r_1, r_2 are real.

The first form is constructed from the line $y=A_0x+A_1$, by the successive operations M_0, which yields $A_0x^2+A_1x$, and ${}^yP_{A_2}$ which gives the required form. This is plainly the graphic analogue of the synthetic substitution in the formation of the table by computation.

The second form is obtained from the line $y=x+p$ by the operations M_0, ${}^yP_q$, ${}^yS_{A_0}$.

The third form comes from the standard parabola $y=x^2$, by means of ${}^xP_a$, which yields $(x-a)^2$, then ${}^yP_b$ which gives $(x-a)^2+b$, then finally ${}^yS_{A_0}$.

The fourth form results from application of M_{r_2} to the line $A_0(x-r_1)$, or of M_{r_1} to the line $A_0(x-r_2)$.

The character of b as negative, 0, or positive, gives the classification of quadratic expressions as having two, one, or no real roots.

There are here four different choices of the set of three constants needed to determine the parabola, sets (A_0, A_1, A_2), (A_0, p, q), (A_0, a, b), (A_0, r_1, r_2). The determination of the connections between these sets affords a fairly extensive exercise in the manipulation of formulas, the complete problem being to express in terms of the constants of any set those of the remaining three.

For the further illustration of the methods of construction for higher degrees, it will perhaps be sufficient to consider a definite case in some detail.

Let the formula be $y=2x^3-4x^2-22x+24$. Alternative constructions are then:

(I) Operate on line $2x-4$, by M_0, giving $2x^2-4x$; then by ${}^yP_{-22}$, giving $2x^2-4x-22$; then by M_0, giving $2x^3-4x^2-22$; then by ${}^yP_{-22}$, (synthetic substitution).

(II) Write the formula $y=2[(x-\frac{2}{3})^3-\frac{29}{3}x+\frac{332}{27}]$. Here give the standard cubic x^3 an x-push of value $\frac{2}{3}$, add the ordinates of the line $-\frac{29}{3}+\frac{332}{27}$, and make y-stretch with factor 2.

(III) The cubic can be factored to $2(x-4)(x-1)(x+3)$. Operate on line $2(x-4)$ by M_1, then by M_{-3}.

The table of the same cubic may have the form:

x	$f(x)$	Δ_1	Δ_2	Δ_3
3.0	-24.00			
		$+7.75$		
3.5	-16.00		8.50	
		$+16.25$		1.50
4.0	0.00		10.00	
		$+26.25$		1.50
4.5	$+26.25$		11.50	
		$+37.75$		1.50
5.0	$+64.00$		13.00	
		$+50.75$		
5.5	114.75			

§4. Algebraic Theorems.

Sufficient experience in construction of graphs belonging to a variety of formulas leads by induction to the formulation of a number of theorems of which a few may be written down here.

If the graph of a rational function meets the x-axis at the point a, after the manner of the elementary function $(x-a)^n$, its algebraic expression contains the factor $(x-a)^n$ in the numerator; if it goes to infinity like the function $(x-a)^{-n}$ it has the factor $(x-a)^{-n}$ in the denominator. These considerations lead to the general factor-theorem, with the notions of simple and multiple roots and asymptotes, and the attempt to represent the given function by *addition* of elementary functions, each with a single asymptote, developes into a study of resolution into partial fractions.

If the slopes of the secant-lines are obtained by division of the first differences in the table by the x-difference, and their values plotted into a set of curves, one for each chosen value of the x-difference, these curves, "secant-slope curves," are seen to approach coincidence with the tangent-slope curve, or "derived curved," which gives the slope of the tangent line as a function of x. Similarly the rth differences divided by the rth power of the x-difference give curves which approach the rth slope-curve, yielding the successive derivatives of the function. This mode of attack is however, in the case of the higher derivatives, convenient mainly for the polynomials.

Inspection of tables computed from polynomials of various degrees suggests that for a polynomial of degree n the first differences act like a function of degree $n-1$, the second like one of degree $n-2$, and so on, the nth differences being all alike, or a function of degree 0. The proof of this is obtained through the agency of the binomial theorem, proved from the theory of combinations or by induction, permitting the expansion of $(x+w)^n$, $(x+2w)^n$, ..., where w is the value of the x-space of the table, and there may be obtained here directly the formulas for the successive derivatives of x^n, together with the theorems that multiplication by a constant multiplies the derivative, and that the derivative of a polynomial is the sum of the derivatives of the separate terms. Then finally the expansion of any polynomial of degree n is seen to take the form of Taylor's theorem for the expansion of $f(x+h)$, but ending with the nth power of h because of the vanishing of the higher derivatives.

The binomial coefficients ("Pascal Triangle") appear also if an error is made in some entry of the table, this error being propogated into the columns of differences, magnified according to the binomial coefficients with alternating signs.

§5. Interpolation Formula. Linear Equations. Determinants.

Problem TF may be solved in a special case in the form: given $n+1$ entries in a table to find a polynomial of degree n yielding those values. One method of solution is to consider the required function as the sum of n polynomials, each of which goes through one of the given points and the projections

of the remaining n points on the x-axis. This yields immediately the classic interpolation formula of LaGrange.

Another solution is to determine the coefficients directly from the linear equations defining the conditions imposed. This introduces the theory of determinants, which is also suggested by the problem of indeterminate coefficients in the partial fraction expansions, and by the problem of finding the intersection of two linear graphs.

§6. Converse Problem. Determination of Roots.

The previous discussion has considered chiefly the problem, given the value of the argument x, to find the value of the rational function $f(x)$. The converse of this, given the value of $f(x)=b$, to find the value or values of x, is immediately seen to be equivalent to the problem, find the roots of the rational function $f(x)-b$, *i. e.*, the roots of the numerator when the function is written as the quotient of two polynomials. The numerical work is carried out by the Horner method of well-directed trial and error, with change of origin and synthetic substitution.

The preceding sketch is plainly rough and incomplete, but is perhaps sufficient to indicate a trend of thought which has been found to yield abundant material for a quarter's work, without sacrifice of unity of structure. Many other topics may be connected easily with those indicated if time permits.

GRAPHICAL METHODS IN TRIGONOMETRY.

By DR. L. E. DICKSON.

Aside from the important work on the solution of triangles by diagrams drawn to scale, graphic methods are not usually employed in trigonometry. Even if the cartesian graphs of the trigonometric functions are constructed, no serious applications are made of these graphs. They are, however, admirably adapted to the explanation of interpolation, to the visualization and retention in the memory of the ratios for the angles 0°, 90°, etc. (in contrast to their derivation as limiting values), and to the natural solution of trigonometric equations,—in particular, to the visualization of the number of angles $<180°$ having a given sine or cosine. In addition to these minor advantages resulting from a frequent appeal to the graphs, the graphic method may be employed to perform the highly important service of leading the student naturally to the majority of the fundamental trigonometric formulae, including the addition theorem and formulae for conversion of sum into product. This is in marked contrast to the current method by which each formula makes its appearance from some unseen source, to be followed by a more or less artificial proof.